**Exam Quantum Field Theory**
**January 20, 2021**
**Start: 14:00h End: 17:00h + 30' for upload**

***Upload your exam solutions (preferably in one .pdf file) to your Dropbox folder and make sure that your name and student number are visible in the uploaded documents.***

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also lose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

For the energy projectors for spin 1/2 Dirac fermions use the normalization without the factor $1/(2m)$:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \not{p} + m$$
$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \not{p} - m$$

$$\{\gamma_5, \gamma^\mu\} = 0, \qquad \left(\gamma^0\right)^2 = \mathbb{1}, \qquad \gamma_5^2 = \mathbb{1}, \qquad \gamma_5^\dagger = \gamma_5, \qquad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$$

$$\gamma^\mu\gamma_\mu = 4\mathbb{1} \quad \gamma^\mu\not{p}\gamma_\mu = -2\not{p} \quad \not{k}\not{p}\not{k} = 2(pk)\not{k} - k^2\not{p}$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

**1.** (3 points total) Consider the Yukawa theory with Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi + g\bar{\psi}\psi\phi \tag{1}$$

which describes the interaction with coupling $g$ of a real scalar field $\phi$ with mass $M$ and a Dirac spinor field $\psi$ with mass $m$.

a) **[2 points]** Given the general formula for the 2-point correlation function for $\phi$:

$$G^{(2)}_\phi(x_1, x_2) = \frac{1}{Z[J=\eta=\bar{\eta}=0, g=0]} \int \mathcal{D}\phi \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\int d^4x\,\mathcal{L}}\, \phi(x_1)\phi(x_2) \tag{2}$$

derive the complete expression (connected and disconnected contributions) for $G^{(2)}_\phi(x_1, x_2)$ in coordinate space up to and including order $g^2$. **Show your work**

b) **[1 point]** Draw the Feynman diagrams corresponding to each contribution in coordinate space.

**2.** (3 points total) Consider QED with a massive photon and a gauge-fixing term. Its Lagrangian density reads:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is antisymmetric in the indices $\mu$ and $\nu$, i.e., $F_{\mu\nu} = -F_{\nu\mu}$, and $\xi$ is an arbitrary gauge-fixing parameter.

a) **[1 point]** According to Noether's theorem, the invariance of the QED Lagrangian density with a massless photon under a $U(1)$ global transformation implies that the current $J_\mu = \bar{\psi}\gamma_\mu\psi$ is conserved, i.e. $\partial_\mu J^\mu = 0$. Explain why this current is still conserved when the photon is massive and the Lagrangian is given by Eq. (3). **Show your work**

b) **[1 point]** Derive the equation of motion (EoM) for the photon field, given the Lagrangian density in Eq. (3). **Show your work**

c) **[1 point]** Taking the divergence of the above EoM, using the antisymmetry of $F_{\mu\nu}$ and that $J_\mu$ is conserved, show that $\partial \cdot A \equiv \partial_\mu A^\mu$ satisfies the equation:

$$\left(\frac{1}{\xi}\Box + M^2\right)\partial \cdot A = 0 \tag{4}$$

which means that $\partial \cdot A$ is a free field. **Show your work**

**Hint:**

- The Euler-Lagrange equation (EoM) for $A_\mu$ reads:

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} = \frac{\delta\mathcal{L}}{\delta A_\nu} \tag{5}$$

**3.** (3 points total) Consider the lagrangian of a Yukawa theory where the real scalar field interacts with a massive spin-1 vector boson (a massive photon) via a term $\phi A_\mu A^\mu$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}M^2\phi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M_A^2 A_\mu A^\mu + f\bar{\psi}\psi\phi + g\phi A_\mu A^\mu \tag{6}$$

The Feynman rules for the vertices are $if$ for the Yukawa interaction and $2ig\, g^{\mu\nu}$ for the scalar-vector interaction.

a) **[2 points]** Calculate the unpolarized squared amplitude $X = \left(\mathcal{A}^\dagger \mathcal{A}\right)_{\text{unpol.}}$ for the process $e^+e^- \to \gamma\gamma$ at the tree level. **Show your work**

b) **[1 point]** Using relativistic kinematics show that $X$ in the Centre of Mass (CM) frame can be written in terms of the initial total energy (or equivalently the Mandelstam variable $s$), the electron mass $m$, the scalar mass $M$, and the vector boson mass $M_A$ only. **Show your work**

**Hint:**

- Use for the sum over the $A_\mu$ polarizations

$$\sum_{a=1}^{3} \epsilon_a^\mu(k)\epsilon_a^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M_A^2} \tag{7}$$